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Short Communication

An analytical investigation of free vibration for a thin-walled regular polygonal prismatic shell with simply supported odd/even number of sides

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1. Introduction

Since thin-walled regular polygonal prismatic shells have higher stiffness of bending resistance compared with solid members with the same material and weight, they have been widely used in various industries such as honeycomb core of sandwich plates, guide support of welding frameworks, and high pier of highway bridges. Therefore, a number of researches including the static buckling, the post-buckling and the dynamic characteristic of these kinds of structures have been performed. Makoto Miyazaki et al. [1] studied the dynamic and static axial deformation processes for an aluminum square tube by an experiment and analysis method. Tesar [2] and Ohga et al. [3] performed the analysis of static and dynamic distortion behavior of thin-walled members on the basis of the transfer matrix method. Migita and Fukumoto [4] investigated the elasto-plastic local buckling behavior of polygonal thin-walled steel sections analytically and experimentally, and this average buckling stresses were examined against width–thickness parameters, aspect ratios, bent angles, the number of sides and the buckling modes. Krajcionovic [5] presented an approximate method for the vibration analysis of prismatic shell with a hexagonal cross-section. Rhodes [6] reviewed the buckling of thin plates and members and early work on rectangular tubes. Wang [7] measured vibration characteristic of the continuous box girder. Kim

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et al. [8,9] researched the spatial free vibration and the spatial stability of nonsymmetric thin-walled curved beams by an energy method. Most of the investigations on prismatic shells were concentrated on the critical loading of static buckling and post-buckling and the distortional vibration, and the methods employed were finite element method (FEM), experimental method and energy method [10–17]. But there are few research reports relating to flexural vibration of a thin-walled regular polygonal prismatic shell, which are foundations of its dynamic analysis and design. In the present study, the novel plate model and beam model of free vibration investigation for a thin-walled regular polygonal prismatic shell simply supported with even number of sides are presented on the basis of its geometrical symmetry. Combining the vibration theories of Euler beam and thin-walled plate, the analytic solutions of natural frequency and mode shape are obtained. The short thin-walled prismatic shell and the long thin-walled prismatic shell have been classified according to this first-order mode shape. It can be found that the different kinds of thin-walled prismatic shells have different mode shapes and different frequency governing equations. The varying curves of natural frequencies of regular polygonal prismatic shells with the number of sides are provided in detail. Finally, 3D finite element numerical simulations have verified the present formulas of natural frequencies and mode shapes. The conclusions will have significant guidelines for the dynamic research of thin-walled prismatic shells.

2. Free vibration of a regular polygonal prismatic shell with even number of sides

The natural frequency and the mode shape of a thin-walled regular polygonal prismatic shell are very complex. According to the superposition principle of mode shape, the dynamic response of structure is a linear combination of every mode shape, and the weight factor of every mode shape has some relations with its modal frequency. The lower the frequency, and the larger the weight factor. In other words, the dynamic behavior of the structure is determined by the low-order modals; thus the low order natural frequencies and their mode shapes are very interesting for the designing analysis. In this paper, more attention is paid to the first-order natural frequencies and their mode shapes varied with geometric parameters.

2.1. Theory

For a thin-walled regular polygonal prismatic shell with even number of sides simply supported on two ends, the assumptions, due to its geometrical symmetry of cross-section, are made as follows:

(1) When the prismatic shell is short, the mode shape of the overall shell is based on a single plate. Every plate has the same mode shape, the phases among the single plates are same or reverse, and the prismatic edges remain a straight line.
(2) If the prismatic shell is long, the mode shape of the whole shell is the deflection of the axial centerline, and the deflection of prismatic edges is the same as the axial centerline.

According to assumption 1, every plate becomes a rectangular thin one simply supported on all four edges, and the boundary condition is described in Fig. 1a (hereafter simply called the plate

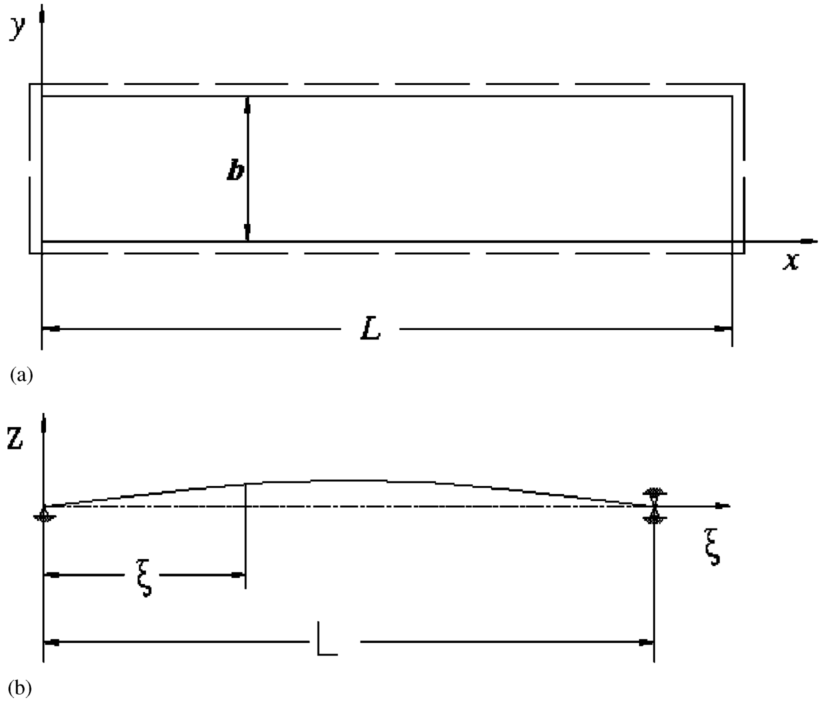


Fig. 1. The mechanics models of free vibration: (a) the rectangle plate simply supported on all four edges; (b) the pin-ended beam.

model). Thus, taking a single plate of the thin-walled regular polygonal prismatic shell for example, the free vibration investigation of the single plate can be substituted for the entire prismatic shell.

For the plate model, the governing equations of free vibration can be expressed as follows:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{1}$$

where w is the displacement perpendicular to the plate surface. h, ρ and t is the plate thickness, material density and time, respectively. D is the flexural stiffness of the single plate, $D = Eh^3/12(1-\mu^2)$. E and μ are Young's modulus and Poisson's ratio, respectively.

On the basis of assumption 2, the free vibration investigation of one beam can be substituted for the entire prismatic shell. Fig. 1b shows the mechanics model of a free vibration beam (hereafter simply called the beam model). The following relationship is given by

$$EJ\frac{\mathrm{d}^4 z}{\mathrm{d}\xi^4}+\rho A\frac{\partial^2 z}{\partial t^2}=0, \tag{2}$$

where z, A and J are the deflection of axial centerline, the cross-section area of the solid part of prismatic shell and the minimum moment of cross-section area against the neutral axis, respectively.

z in Eq. (2) and w in Eq. (1) are not quite the same in the physical meanings; the former expresses the axial centerline deflection of the whole prismatic shell, and the latter shows the displacement perpendicular to the single plate.

If the length of the prismatic shell is short, the solution (mode shape) of Eq. (1) is expressed as follows:

$$w = A_{mn}\ \sin\frac{m\pi x}{L}\sin\frac{n\pi y}{b}\sin(\omega t - \phi_1), \tag{3}$$

where $m,n = 1,2,3,\ldots,\ \infty$. L and b are the length of prismatic shell and the width of a single plate, respectively. A_{mn} is the nontrivial factor. ω and ϕ_1 are the natural frequency of prismatic shell and initial phase of a single plate vibration, respectively.

The analysis solution of natural frequency for the thin-walled regular polygonal prismatic shells with even number of sides can be derived from Eqs. (1) and (3)

$$\omega_{mn} = \pi^2\left(\frac{m^2}{L^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho h}}. \tag{4}$$

Then, the lowest value of ω_{11} is given when n, m are taken as 1.

$$\omega_{11} = \pi^2\left(\frac{1}{L^2} + \frac{1}{b^2}\right)\sqrt{\frac{D}{\rho h}}. \tag{5}$$

For the beam model 1b, the solution (mode shape) of Eq. (2) can then be obtained in the following form

$$z(\xi, t) = c_k \sin\frac{k\pi\xi}{L}\sin\omega t, \tag{6}$$

where $k = 1,2,3,\ldots,\ \infty$. c_k is a nontrivial factor. The natural frequency ω_k of the beam model will be derived from Eqs. (2) and (6)

$$\omega_k = \frac{k^2\pi^2}{L^2}\sqrt{\frac{JE}{\rho A}}. \tag{7}$$

The lowest value of ω_1 can be obtained when k is taken as 1

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EJ}{\rho A}}. \tag{8}$$

For a thin-walled regular polygonal prismatic shell with even number of sides, Eqs. (4) and (7) are analysis solutions of natural frequencies of plate model and beam model, respectively; Eqs. (3) and (6) are analysis solutions of mode shapes of plate model and beam model, respectively.

2.2. Classifying

After acquiring the expressions of natural frequency and mode shape, the short prismatic shell and the long prismatic shell will be classified according to the first-order mode shape. It can be found out that different kinds of prismatic shells have different frequency governing equations.

For a given b, m, n and k, the natural frequencies of prismatic shells in Eqs. (4) and (7) are increased as the L becomes shorter, and the latter is faster. If L is very long, the limits of natural frequency in Eqs. (4) and (7) can be obtained as follows:

$$\omega_{mn} = \lim_{L\to\infty} \pi^2\left(\frac{m^2}{L^2}+\frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho h}} = \frac{\pi^2 n^2}{b^2}\sqrt{\frac{D}{\rho h}}, \tag{9}$$

for Eq. (7)

$$\omega_i = \lim_{L\to\infty} \frac{k^2\pi^2}{L^2}\sqrt{\frac{JE}{\rho A}} = 0. \tag{10}$$

Eqs. (9) and (10) illustrate that the natural frequency in Eq. (7) is reduced faster than that in Eq. (4) as the L becomes longer; thus, the point of intersection L_0 must exist between them. Here let m, n, $k=1$, and the equation can be denoted as follows:

$$\frac{\pi^2}{L_0^2}\sqrt{\frac{EJ}{\rho A}} = \pi^2\left(\frac{1}{L_0^2}+\frac{1}{b^2}\right)\sqrt{\frac{D}{\rho h}}, \tag{11}$$

Substitution of this expression for $D = Eh^3/12(1-\mu^2)$ into Eq. (11) yields the equation

$$d = \sqrt{12(1-\mu^2)}\frac{i}{h} - 1, \tag{12}$$

where i is the inertia radius of cross-section, $i = J/A$. $d = L_0^2/b^2$.

The L_0 is as follows:

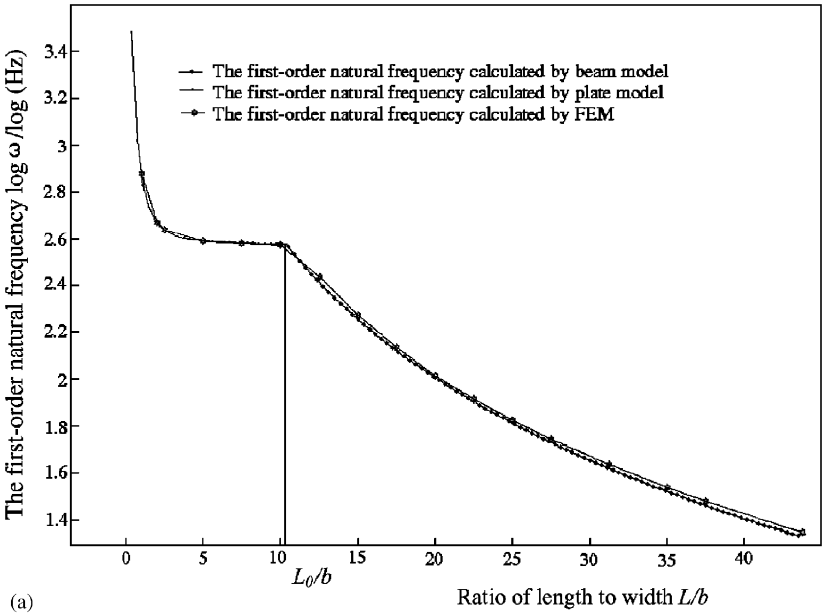
$$L_0 = b\sqrt{d}, \tag{13}$$

Eqs. (12) and (13) show that L_0 is mainly determined by geometrical parameters of the prismatic shell, and it has few relations with the property of material (generally μ is very little). When L is in the range of $L \leqslant L_0$, the corresponding prismatic shell is defined as a short prismatic shell, the natural frequency can be calculated by governing Eq. (4), and the mode shape of a single plate is expressed by Eq. (3). When L is in the range of $L_0 < L$, the corresponding prismatic shell is called a long prismatic shell, the natural frequency can be calculated by governing Eq. (7), and the mode shape of the overall prismatic shell is denoted by Eq. (6).

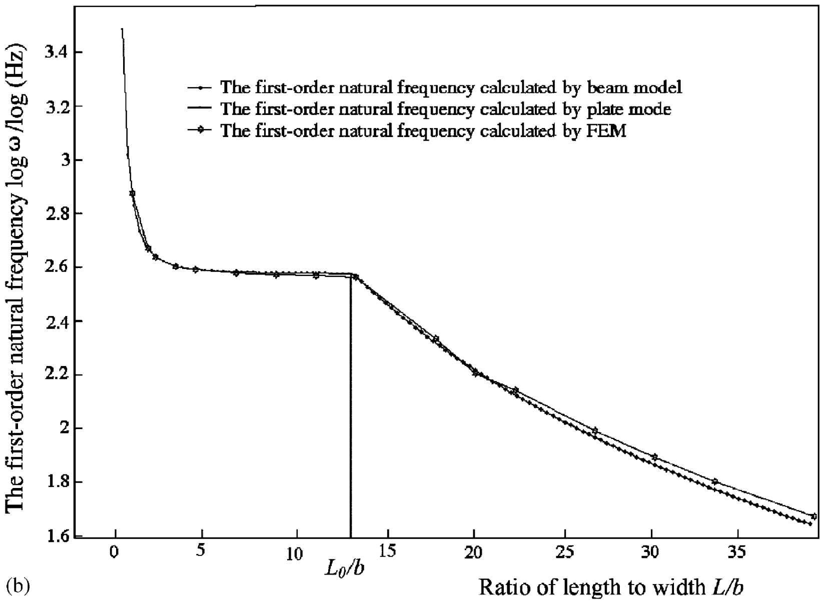
2.3. Numerical example

The square and regular hexagonal prismatic shells (geometric parameters $b = 80$ mm, $t = 1$ mm) made from aluminum (material parameters $E = 6.897 \times 10$ GPa, $\mu = 0.3$) are used in order to verify analytical solutions of natural frequency for a regular polygonal thin-walled prismatic shell with even number of sides. The first-order natural frequencies calculated by Eqs. (5) and (8) are compared with the 3D finite element results. Firstly, classify the prismatic shells according to Eq. (13), and find out the square prismatic shells $L_{0s} = 828.6$ mm and the hexagonal prismatic shells $L_{0h} = 1038.2$ mm. The results are illustrated in Fig. 2. Here the horizontal axis is L/b, and the vertical axis is the logarithm of the natural frequency of the prismatic shell.

Fig. 2 indicates that the results calculated by the present formulas are in good agreement with the 3D finite element simulations, which demonstrates that analytical solution in the present study is valid.

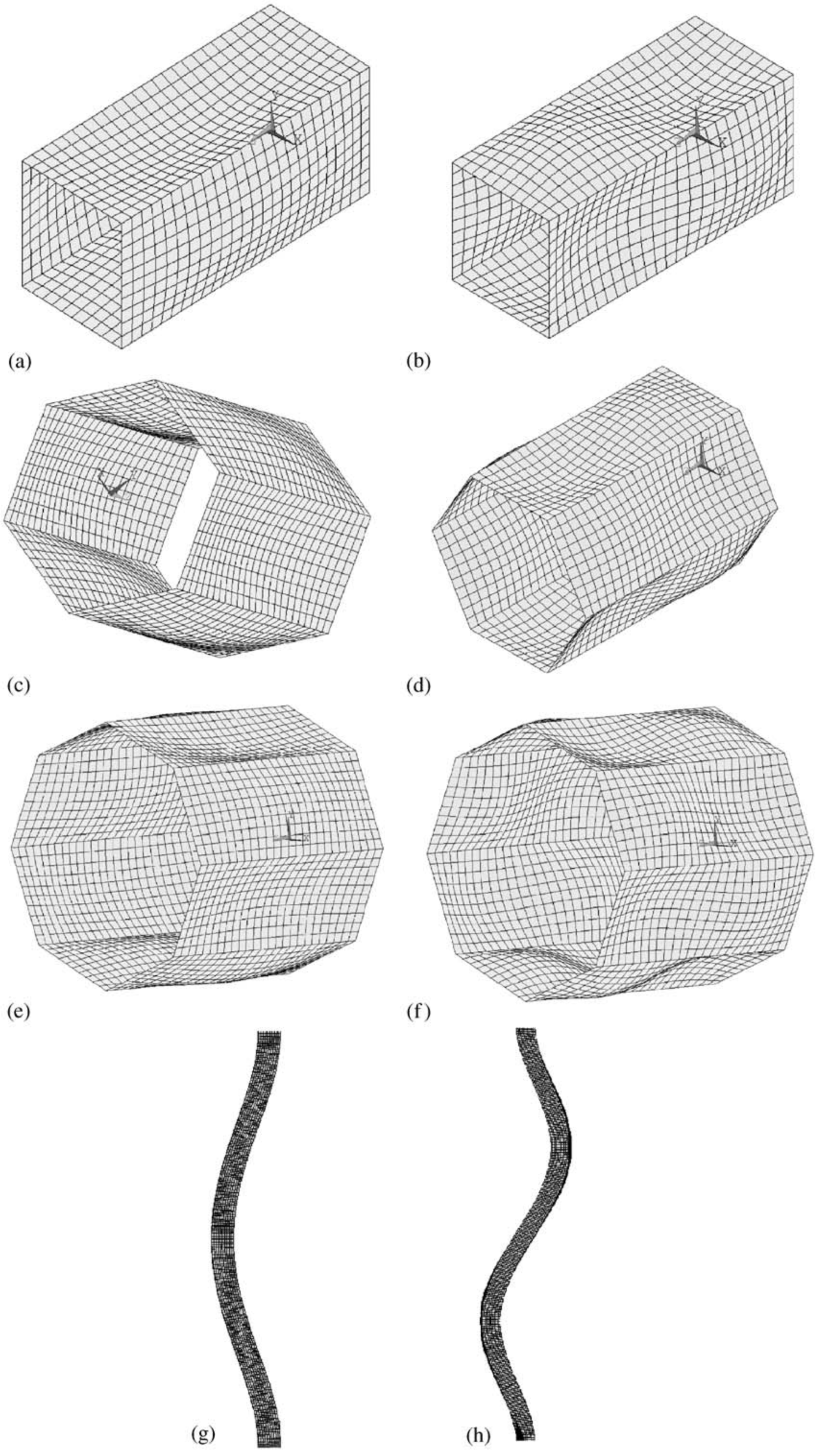


(a)



(b)

Fig. 2. The first-order natural frequencies calculated by present formulas compared with FEM analysis: (a) square prismatic shell; (b) regular hexagonal prismatic shell.


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

In order to verify Eqs. (3) and (6) of mode shapes, the partial modals of free vibration for the thin-walled regular polygonal prismatic shells with even number of sides are packed up according to finite element simulation. The material constants and the geometric parameters of prismatic shells are as mentioned above. Figs. 3a, c and e are first-order mode shapes of the short square, hexagonal, and octagonal prismatic shells, respectively; and Figs. 3b, d and f are second-order mode shapes of the short square, hexagonal, and octagonal prismatic shells, respectively; Figs. 3g and h are first and second-order mode shapes for a long square prismatic shell, respectively. Fig. 3 explains that every plate of a short prismatic shell has the same mode shape as a thin-walled rectangle plate simply supported on all four edges, and a long prismatic shell has the same mode shape as a pin-ended beam. The results indicate that assumptions 1 and 2 are correct.

3. Free vibration of regular polygonal prismatic shell with odd number of sides

For a thin-walled regular polygonal prismatic shell with even number of sides, the restrictions among the single plates are consistent according to the geometrical symmetry of cross-section, which makes every plate have the same mode shape and the prismatic edges remain a straight line, whereas for a thin-walled regular polygonal prismatic shell with odd number of sides, the restrictions among the single plates are not completely uniform due to the uniaxial symmetry of cross-section, which makes every plate have different mode shape and the prismatic edges not to remain a straight line (see the mode shape of cross-section in Fig. 4).

In order to reveal the relationships of first-order natural frequency vs. the number of sides, the regular polygonal prismatic shells made from aluminum are employed. The geometric parameters are $b = 80\,\mathrm{mm}$, $t = 1\,\mathrm{mm}$, $h = 200\,\mathrm{mm}$, and the constants of material are as mentioned above. The results calculated by Eq. (5) are compared with 3D FEM numerical simulations. The varying curves of first-order natural frequency vs. the number of sides are described as Fig. 5 shows. The horizontal axis is the number of sides, and the vertical axis is the first-order natural frequency of the prismatic shell.

Fig. 5 demonstrates that the first-order natural frequency calculated by the plate model is in good agreement with 3D FEM analysis when the number of sides for a thin-walled regular polygonal prismatic shell is even and is less than 40. When the number of sides is odd and is in the range 11–41, the first-order natural frequency used present plate model is basically consistent with 3D finite element results also. If the number of sides is beyond the ranges as mentioned above, there are some errors between the present plate model and 3D finite element results. In this example, the errors of the regular triangular, pentagonal, heptagonal and enneagon prismatic shell are 21.96%, 8.40%, 4.13% and 2.57%, respectively. The result indicates that the

Fig. 3. The partial modals for regular polygonal prismatic shells: (a) $m, n = 1$, the first-order modal of square prismatic shell; (b) $m = 2$, $n = 1$, the second-order modal of square prismatic shell; (c) $m, n = 1$, the first-order modal of hexagonal prismatic shell; (d) $m = 2$, $n = 1$, the second-order modal of hexagonal prismatic shell; (e) $m, n = 1$, the first-order modal of octagonal prismatic shell; (f) $m = 2$, $n = 1$, the second-order modal of octagonal prismatic shell; (g) $k = 1$, the first-order modal of square prismatic shell; (h) $k = 2$, the second-order modal of square prismatic shell.

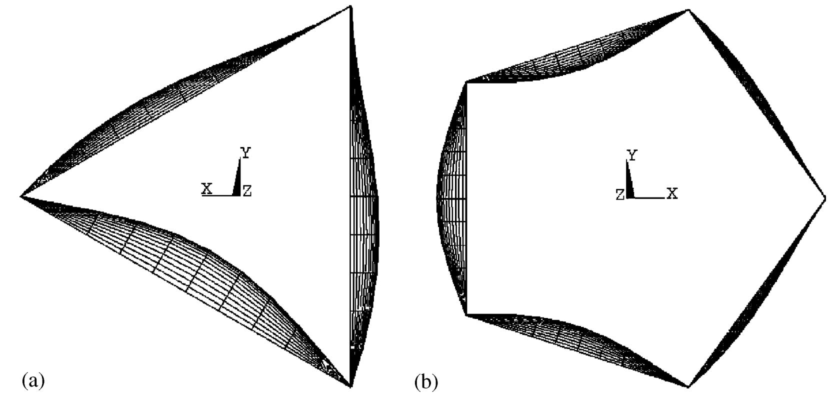


Fig. 4. The partial modals of regular polygonal prismatic shells with odd number of sides: (a) the first-order modal of triangular prismatic shell; (b) the first-order modal of pentagonal prismatic shell.

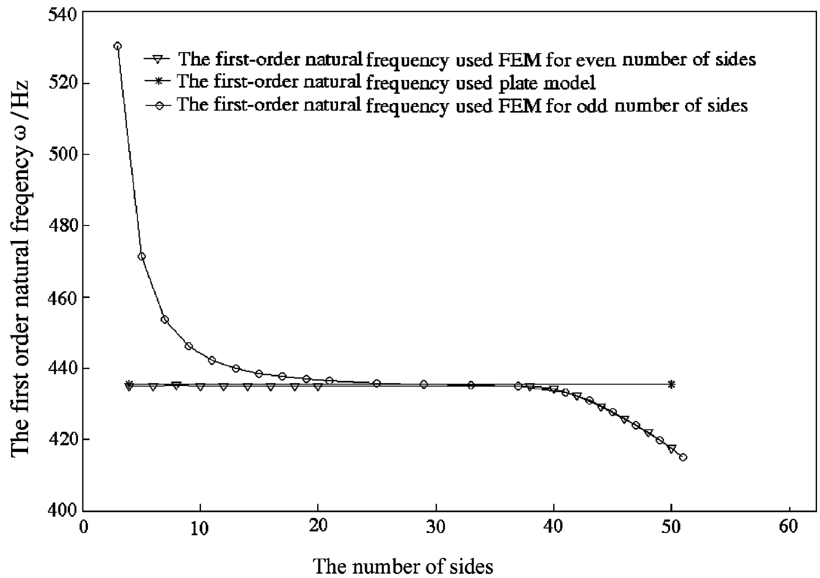


Fig. 5. First-order natural frequency vs. the number of sides.

characteristic of geometric asymmetry is declined as the number of sides increases. When the number of sides is larger than 20, the natural frequency curves of thin-walled regular polygonal prismatic shells both with even number of sides and with odd number of sides become uniform. If the number of sides is larger than 80, the natural frequency of a thin-walled regular polygonal prismatic shell both with even number of sides and with odd number of sides is the same as the natural frequency of the circumcircle column shell of the corresponding prismatic shell. Luckily the number of sides of prismatic shells applicable to engineering is not large (commonly less than 40), and generally has even number of sides.

4. Conclusions

(1) With the vibration theories of the Euler beam and thin-walled plate combined, the plate model and the beam model of free vibration investigation for a thin-walled regular polygonal prismatic shell with even number of sides are presented. The analytic solutions of natural frequency and mode shape are obtained on the basis of its geometrical symmetry of cross-section. The short thin-walled prismatic shells and long thin-walled prismatic shells have been classified according to their first-order mode shape. The perfectly designing theory of dynamic characteristic for the thin-walled regular polygonal prismatic shells is built up. The result calculated by present formulas is coincident with 3D finite element analysis. The results show that the plate model and the beam model reveal the physical reality of the thin-walled regular polygonal prismatic shell.

(2) By the 3D finite element numerical simulation for a thin-walled regular polygonal prismatic shell made from aluminum with even number of sides, the validity of the present formulas of natural frequencies and mode shapes is confirmed, and the correctness of the assumptions proposed by this paper is verified.

(3) For the thin-walled regular polygonal prismatic shells with odd number of sides, when the number of sides is in the range 11–41, the first-order natural frequency calculated by the present plate model is basically consistent with 3D finite element simulation. If the number of sides is beyond the range, there are some errors between the present plate model and 3D finite element result.

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